

Arithmetic properties of generalized Fibonacci sequences

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Abstract

The *generalized* Fibonacci sequences are sequences $\{f_n\}$ which satisfy the recurrence $f_n(s, t) = sf_{n-1}(s, t) + tf_{n-2}(s, t)$ ($s, t \in \mathbb{Z}$) with initial conditions $f_0(s, t) = 0$ and $f_1(s, t) = 1$. In a recent paper, Amdeberhan, Chen, Moll, and Sagan considered some arithmetic properties of the generalized Fibonacci sequence. Specifically, they considered the behavior of analogues of the p -adic valuation and the Riemann zeta function. In this paper, we resolve some conjectures which they raised relating to these topics. We also consider the rank modulo n in more depth and find an interpretation of the rank in terms of the order of an element in the multiplicative group of a finite field when n is an odd prime. Finally, we study the distribution of the rank over different values of s when $t = -1$ and suggest directions for further study involving the rank modulo prime powers of generalized Fibonacci sequences.

1 Introduction

Recall that the Fibonacci sequence $\{F_n\}$ is defined as the sequence satisfying the recurrence $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = 1$. The terms of this sequence satisfy some interesting divisibility properties, which we list below.

Theorem 1 (Vorob'ev). [12] $\gcd(F_n, F_m) = F_{\gcd(n, m)}$ for all $m, n \in \mathbb{Z}^+$. Note that this implies that $\gcd(F_n, F_{n+1}) = 1$.

Theorem 2. [12] If $n \geq 2$, $F_n | F_m$ if and only if $n | m$.

Theorem 3 (Hoggatt and Long). [5] F_n is prime only if n is prime.

Theorem 4. [5] Let $r \in \mathbb{Z}^+$. Then there exists an m such that $r | F_m$. If m is the least such number, then $r | F_n$ if and only if $m | n$.

Many of these divisibility properties are also shared in the following generalization of the Fibonacci sequence.

Definition 1 (ACMS). [1] *The generalized Fibonacci polynomials are polynomials in s and t defined by $f_0(s, t) = 0$ and $f_1(s, t) = 1$ with the recurrence $f_n(s, t) = sf_{n-1}(s, t) + tf_{n-2}(s, t)$ for $n \geq 2$.*

Here are some counterparts of the results above for generalized Fibonacci polynomials. Note that divisibility is considered over $\mathbb{Z}[s, t]$ unless otherwise indicated.

Theorem 5. [5] $\gcd(f_n, f_m) = f_{\gcd(n, m)}$ for all $m, n \in \mathbb{Z}^+$. Note that this implies that $\gcd(f_n, f_{n+1}) = 1$.

Theorem 6. [5] If $n \geq 2$, $f_n | f_m$ if and only if $n | m$.

Theorem 7. [5] $f_n(s, t)$ is irreducible over \mathbb{Q} if and only if n is prime.

Theorem 8. [5] Fix $s, t \in \mathbb{Z}$ and let r be a positive integer with $\gcd(r, t) = 1$. Then there exists m such that $r | f_m$. If there is a least positive integer m such that $r | f_m$, then $r | f_n$ if and only if $m | n$.

The remainder of this paper will mainly deal with generalized Fibonacci sequences, which are the sequences obtained after fixing $s, t \in \mathbb{Z}$ in the generalized Fibonacci polynomials. In the next section, we will consider the periodicity of generalized Fibonacci sequences modulo n in relation to a generalization of p -adic valuations. In Section 3, we consider an analogue of the Riemann zeta function for generalized Fibonacci sequences. Finally, we will examine the rank modulo n in more depth and find an interpretation as the order of an element in a finite field.

2 d -adic valuations of generalized Fibonacci sequences

One property of the Fibonacci sequence which carries over to generalized Fibonacci sequences is periodicity modulo n for any $n \geq 2$. This property also holds for generalized Fibonacci sequences since there are finitely many choices for pairs of consecutive terms modulo n . According to [14], many arithmetic properties of generalized Fibonacci sequences ultimately depend on this periodicity property and the divisibility property which was shown in Theorem 2 and Theorem 6. We now formally define the period modulo m .

Definition 2. The period modulo m is defined to be the smallest positive integer $k(m)$ such that $f_n(s, t) \equiv f_{n+k(m)}(s, t) \pmod{m}$ for all sufficiently large n .

There is a quantity similar to the period related to divisibility modulo m .

Definition 3. If there exists $r(m)$ such that $f_y(s, t) \equiv 0 \pmod{m}$ if and only if $r(m) | y$, $r(m)$ is defined as the rank modulo m .

For example, we can consider the Fibonacci sequence modulo 8. Here are the first few terms of the residues modulo 8: 0, 1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1, 0, 1, 1, ... In this case, the period $k(8) = 12$, while the rank $r(8) = 6$ [11].

We now show that the rank exists modulo a prime p if and only if $p \nmid t$.

Theorem 9. *Let p be a prime. Then, $r(p)$ exists if and only if $p \nmid t$.*

Proof. By Theorem 8, there is a positive integer m such that $p \mid f_m$ if $p \nmid t$. This means that it suffices to show that it is not possible to obtain a sequence of the form $x, 0, 0, \dots$ modulo p with $x \neq 0$ in this case. Using the recurrence, we find that this is only possible if there is a t such that $tx \equiv 0 \pmod{p}$. If $p \mid s$ and $p \mid t$, then this is true for any x and $f_n \equiv 0 \pmod{p}$ for all $n \geq m$, so $r(p)$ does not exist. If $p \nmid s$ and $p \mid t$, the sequence would be of the form $0, 1, s, s^2, s^3, \dots \pmod{p}$ and there would be no positive integer m such that $p \mid f_m$. If $p \nmid t$, this statement implies that $x \equiv 0 \pmod{p}$. However, this contradicts the assumption that $x \not\equiv 0 \pmod{p}$. So, the pair (f_m, f_{m+1}) must be of the form $(0, a)$ modulo p , where $a \neq 0$ and the sequence up to f_{m-1} is multiplied by $a \pmod{p}$. Then, $r(p)$ is equal to the smallest possible value of m . \square

Arguments similar to the one used above can be used to show that $r(p^k)$ exists if and only if $r(p)$ exists as p^k is relatively prime to t if and only if $p \nmid t$. We will mainly focus on the case where $t = -1$ since $r(m)$ exists for any $m \geq 2$ and many properties of the original Fibonacci sequence carry over to this case.

Here are some results on the rank of generalized Fibonacci sequences modulo p , where p is an odd prime.

Theorem 10 (Li). [8] *Let p be an odd prime and $r(p)$ be the rank of the generalized Fibonacci sequence with parameters $(s, t) \pmod{p}$. Let $D = s^2 + 4t$. If $p \mid D$, we have that $r(p) = p$.*

Theorem 11. [8] *If p is an odd prime such that $p \nmid D$, then $(-t/p) = 1$ if and only if $r(p) \mid \frac{p - (-D/p)}{2}$.*

Remark 1. *Note that we always have $(-t/p) = 1$ when $t = -1$. By Theorem 10 and Theorem 11, we have that $r(p) \leq p$ for all odd primes p .*

Another property of the Fibonacci sequence which carries over to generalized Fibonacci sequences with $t = -1$ is that we can relate periods of generalized Fibonacci sequences mod p to p -adic valuations of their elements. This is one example of the relationship of periodicity modulo n to other arithmetic properties of generalized Fibonacci sequences. Here are some results which have similar counterparts for the Fibonacci sequence (see [7], [13], [3], and [7] for Lemmas 1, 2, 3, and 4 respectively). We use $g_n(s, t)$ to denote the generalized Lucas sequences.

Lemma 1. $\gcd(f_n(s, -1), g_n(s, -1)) \leq 2$.

Proof.

$$\begin{aligned}
\gcd(f_n(s, -1), g_n(s, -1)) &= \gcd(f_n(s, -1), f_{n+1}(s, -1) - f_{n-1}(s, -1)) \text{ (Proposition 2.3 of [1])} \\
&= \gcd(f_n(s, -1), sf_n(s, -1) - 2f_{n-1}(s, -1)) \\
&= \gcd(f_n(s, -1), -2f_{n-1}(s, -1)) \\
&= \gcd(f_n(s, -1), 2) \text{ (follows by induction)}
\end{aligned}$$

□

Lemma 2. $f_{kn}(s, -1) = 2^{1-k} f_n(s, -1)(Kf_n(s, -1)^2 + kg_n(s, -1)^{k-1})$, where $K \in \mathbb{Z}$.

Proof. Consider the binomial expansion of $f_{kn}(s, -1) = \frac{X^{kn} - Y^{kn}}{X - Y}$, where $X = \frac{s + \sqrt{s^2 - 4}}{2}$ and $Y = \frac{s - \sqrt{s^2 - 4}}{2}$.

$$\begin{aligned}
f_{kn}(s, -1) &= \frac{X^{kn} - Y^{kn}}{X - Y} \\
&= \frac{1}{X - Y} \left(\frac{1}{2^k} ((X - Y)f_n(s, -1) + g_n(s, -1))^k - \frac{1}{2^k} (-(X - Y)f_n(s, -1) + g_n(s, -1))^k \right) \\
&= 2^{1-k} \sum_{j \text{ odd}} \binom{k}{j} (X - Y)^{j-1} f_n(s, -1)^j g_n(s, -1)^{k-j} \\
&= 2^{1-k} f_n(s, -1)(Kf_n(s, -1)^2 + kg_n(s, -1)^{k-1}) \text{ (where } K \in \mathbb{Z})
\end{aligned}$$

□

Lemma 3. Let $k \in \mathbb{N}$. We have $f_{kn}(s, -1) \equiv kf_{n+1}(s, -1)^{k-1}f_n(s, -1) \pmod{f_n(s, -1)^2}$ and $f_{kn+1}(s, -1) \equiv f_{n+1}(s, -1)^k \pmod{f_n(s, -1)^2}$.

Proof. We use induction on k . If $k = 1$, we have $f_n(s, -1) \equiv f_n(s, -1) \pmod{f_n(s, -1)^2}$ and $f_{n+1}(s, -1) \equiv f_{n+1}(s, -1) \pmod{f_n(s, -1)^2}$. Assume that $f_{kn}(s, -1) \equiv kf_{n+1}(s, -1)^{k-1}f_n(s, -1) \pmod{f_n(s, -1)^2}$ and $f_{kn+1}(s, -1) \equiv f_{n+1}(s, -1)^k \pmod{f_n(s, -1)^2}$. Then, we have the following.

$$\begin{aligned}
f_{(k+1)n}(s, -1) &= f_{kn+1}(s, -1)f_n(s, -1) - f_{kn}(s, -1)f_{n-1}(s, -1) \text{ (Theorem 2.2 of [1])} \\
&\equiv f_{n+1}(s, -1)^k f_n(s, -1) - kf_{n+1}(s, -1)^{k-1} f_n(s, -1)f_{n-1}(s, -1) \pmod{f_n(s, -1)^2} \\
&\equiv f_{n+1}(s, -1)^k f_n(s, -1) + kf_{n+1}(s, -1)^{k-1} f_n(s, -1)f_{n+1}(s, -1) \pmod{f_n(s, -1)^2} \\
&\equiv (k+1)f_{n+1}(s, -1)^k f_n(s, -1) \pmod{f_n(s, -1)^2}
\end{aligned}$$

$$\begin{aligned}
f_{(k+1)n+1}(s, -1) &= f_{kn+1}(s, -1)f_{n+1}(s, -1) - f_{kn}(s, -1)f_n(s, -1) \\
&\equiv f_{n+1}(s, -1)^k f_{n+1}(s, -1) - kf_{n+1}(s, -1)^{k-1} f_n(s, -1)^2 \pmod{f_n(s, -1)^2} \\
&\equiv f_{n+1}(s, -1)^{k+1} \pmod{f_n(s, -1)^2}
\end{aligned}$$

□

Lemma 4. *Let p be an odd prime and $e = e(p) = \nu_p(f_{r(p)}(s, -1))$. Then,*

$$\nu_p(f_n(s, -1)) = \begin{cases} \nu_p(n) - \nu_p(r(p)) + e(p), & \text{if } n \equiv 0 \pmod{r(p)}. \\ 0, & \text{if } n \not\equiv 0 \pmod{r(p)}. \end{cases}$$

Proof. By Lemma 2, we have that $f_{kn}(s, -1) = 2^{1-k} f_n(s, -1) (K f_n(s, -1)^2 + k g_n(s, -1)^{k-1})$, where $K \in \mathbb{Z}$. Let $k = p$, $n = cr(p)p^{\alpha-1}$, where $\gcd(c, p) = 1$ and $\alpha \geq 1$. This means that $f_{cr(p)p^\alpha}(s, -1) = 2^{1-p} f_{cr(p)p^{\alpha-1}}(s, -1) (K' p^2 + p g_{cr(p)p^{\alpha-1}}(s, -1)^{p-1})$ ($K' \in \mathbb{Z}$). Note that $p \nmid g_{cr(p)p^{\alpha-1}}(s, -1)$ since $\gcd(f_n(s, -1), g_n(s, -1)) \leq 2$ by Lemma 1 and p is an odd prime. This implies that $\nu_p(f_{cr(p)p^\alpha}(s, -1)) = \nu_p(f_{cr(p)p^{\alpha-1}}(s, -1)) + 1$ and it follows by induction that $\nu_p(f_{cr(p)p^\alpha}(s, -1)) = \nu_p(f_{cr(p)}(s, -1)) + \alpha$.

We also have by Lemma 3 that $f_{cr(p)}(s, -1) \equiv c f_{r(p)}(s, -1) f_{r(p)+1}(s, -1)^{c-1} \pmod{p^{2e}}$. Note that $\gcd(f_{r(p)+1}(s, -1), p) = 1$ since $f_m(s, -1) \equiv 0 \pmod{p}$ if and only if $r(p) \mid m$. Since $r(p) \nmid r(p) + 1$, we have $p \nmid f_{r(p)+1}(s, -1) \Rightarrow \gcd(f_{r(p)+1}(s, -1), p) = 1$ since p is prime. This implies that $\nu_p(f_{cr(p)}(s, -1)) = \nu_p(f_{r(p)}(s, -1))$ when $\gcd(c, p) = 1$. This can be used with $\nu_p(f_{r(p)}(s, -1)) = e < 2e$ to show that $\nu_p(f_{cr(p)p^\alpha}(s, -1)) = \nu_p(f_{r(p)}(s, -1)) + \alpha = e(p) + \alpha$. Thus, $\nu_p(f_n(s, -1)) = \nu_p(n) - \nu_p(r(p)) + e(p)$ when $r(p) \mid n$. \square

Remark 2. *It would be interesting to generalize this lemma to relate p -adic valuations to the rank modulo p for other values of t .*

The results above relating the period of the generalized Fibonacci sequence to the p -adic valuations of its elements can be used to show that the following conjecture of Amdeberhan, Chen, Moll, and Sagan holds.

Conjecture 1. *[1] Suppose $s \geq 2$ is an integer and $d \geq 3$ is an odd integer. There exist integers $s'(s, d)$ and $d'(s, d)$ such that $d' \leq d$ and $\nu_d(f_n(s, -1)) = \delta_{d'\mathbb{Z}}(n) \nu_d(\frac{s'n}{d'})$.*

We will first show that this conjecture holds when d is an odd prime.

Theorem 12. *Suppose $s \geq 2$ is an integer and $p \geq 3$ is an odd prime. There exist integers $s'(s, d), d'(s, d)$ such that $d' \leq d$ and $\nu_p(f_n(s, -1)) = \delta_{d'\mathbb{Z}}(n) \nu_p(\frac{s'n}{d'})$.*

Proof. If there exists d' such that if $\delta_{d'\mathbb{Z}}(n) = 0$, we have that $\nu_d(f_n(s, -1)) = 0$. This implies that $d' \nmid n \Rightarrow d \nmid f_n(s, -1)$, which is equivalent to the statement $d \mid f_n(s, -1) \Rightarrow d' \mid n$. Note that d' divides the rank mod p . Let $r(p)$ be the rank mod p . Setting $d' = r(p)$ satisfies the previous condition and we have $d' \leq p$ (see remark following Theorem 4).

Consider the case where $r(p) \mid n$. Then, $\nu_p(f_n(s, -1)) = \nu_p(\frac{s'n}{d'})$ if and only if $\nu_p(n) - \nu_p(r(p)) + e(p) = \nu_p(\frac{s'n}{d'})$. If we take s' such that $d' \mid s'$, we have $\nu_p(\frac{s'n}{d'}) = \nu_p(n) + \nu_p(\frac{s'}{d'})$ and the statement reduces to $\nu_p(\frac{s'}{d'}) = -\nu_p(r(p)) + e(p)$. Note that

we either have $\nu_p(r(p))$ equal to 0 or 1, which means that $-\nu_p(r(p)) + e(p) \geq 0$ as $p \mid f_{r(p)}(s, -1)$. This equation is satisfied if we set $s' = d' \cdot p^{-\nu_p(r(p)) + e(p)}$, which only depends on s and p . \square

This result can be generalized to the case where d is the power of an odd prime.

Theorem 13. *Suppose $s \geq 2$ is an integer and $d = p^r$, where p is an odd prime ($p \geq 3$). There exist integers $s'(s, d)$ and $d'(s, d)$ such that $d' \leq d$ and $\nu_d(f_n(s, -1)) = \delta_{d'} \mathbb{Z}(n) \nu_d(\frac{s'n}{d'})$.*

Proof. We set d' equal to some divisor of $r(d)$ less than or equal to d (eg. $d' = 1$). We now consider the case where $r(d) \mid n$. We claim that there exists an s' divisible by d' such that $\nu_d(f_n(s, -1)) = \nu_d(\frac{s'n}{d'})$. Since $\nu_{p^r}(N) = \lfloor \frac{\nu_p(N)}{r} \rfloor$, we want to find s' such that $\lfloor \frac{\nu_p(n) - \nu_p(r(p)) + e(p)}{r} \rfloor = \lfloor \frac{\nu_p(\frac{s'n}{d'})}{r} \rfloor = \lfloor \frac{-\nu_p(n) + \nu_p(\frac{s'}{d'})}{r} \rfloor$. We find that setting $s' = d' \cdot p^{-\nu_p(r(p)) + e(p)}$ as in the end of the proof of Theorem 12 also satisfies this equation since the numerators are equal. \square

We can use solutions for powers of odd primes to show that the conjecture holds for arbitrary odd integers d .

Theorem 14. *Suppose $s \geq 2$ is an integer and $d \geq 3$ is an odd integer. There exist integers $s'(s, d)$ and $d'(s, d)$ such that $d' \leq d$ and $\nu_d(f_n(s, -1)) = \delta_{d'} \mathbb{Z}(n) \nu_d(\frac{s'n}{d'})$.*

Proof. Set d' to be a divisor of the rank mod d less than or equal to d as above.

Let $d = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ and $s' = d' \cdot \prod_{j=1}^k p_j^{-\nu_{p_j}(r(p_i)) + e(p_j)}$. We have $\nu_{p_i}(n) = \nu_{p_i}(n) - \nu_{p_i}(r(p_i)) + e(p_i)$ and $\nu_{p_i}(\frac{s'n}{d'}) = \nu_{p_i}(n) + \nu_{p_i}(\frac{s'}{d'}) = \nu_{p_i}(n) + \nu_{p_i}(\prod_{j=1}^k p_j^{\nu_{p_j}(-r(p_j)) + e(p_j)}) = \nu_{p_i}(n) - \nu_{p_i}(r(p_i)) + e(p_i)$. So, we have $\nu_{p_i}(f_n(s, -1)) = \nu_{p_i}(\frac{s'n}{d'})$ for all i . Since $\nu_d(N) = \min_{1 \leq i \leq k} \lfloor \frac{\nu_{p_i}(N)}{\alpha_i} \rfloor$ for $N \in \mathbb{N}$, we have $\nu_d(f_n(s, -1)) = \min_{1 \leq i \leq k} \lfloor \frac{\nu_{p_i}(f_n(s, -1))}{\alpha_i} \rfloor = \min_{1 \leq i \leq k} \lfloor \frac{\nu_{p_i}(\frac{s'n}{d'})}{\alpha_i} \rfloor = \nu_d(\frac{s'n}{d'})$. \square

3 An analogue of the Riemann zeta function

We now consider an analogue of the Riemann zeta function. The Riemann zeta function is defined as the analytic continuation of $\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z}$, where $z \in \mathbb{C}$. One variation of this function which has been considered is the function $\zeta_F(z) = \sum_{k=0}^{\infty} \frac{1}{F_k^z}$ with $z \in \mathbb{C}$, where the positive integers are replaced with terms of the Fibonacci sequence [15]. This function shares some properties with the original Riemann zeta function (see [9] for more details) and has been

studied in several different ways (see [6] for an overview). Some of the work that has been done with this analogue of the Riemann zeta function involves estimating the tails of the series for positive integers z . The first result relating to such estimates is the following.

Theorem 15 (Ohtsuka and Nakamura). [10] *Let $\{F_k\}$ be the Fibonacci sequence.*

$$\left[\left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right] = \begin{cases} F_{n-2}, & \text{if } n \text{ even, } n \geq 2 \\ F_{n-2} - 1, & \text{if } n \text{ odd, } n \geq 1 \end{cases}$$

$$\left[\left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right] = \begin{cases} F_{n-1}F_n - 1, & \text{if } n \text{ even, } n \geq 2 \\ F_{n-1}F_n, & \text{if } n \text{ odd, } n \geq 1 \end{cases}$$

Note that no closed form is known for the sum $\sum_{k=n}^{\infty} \frac{1}{F_k}$ although some of its properties are known. Holliday and Komatsu [6] gave the first result relating to sums of more general terms. Specifically, they studied the case where the denominators were Fibonacci polynomials with a fixed integer parameter. Here is one of their results.

Theorem 16 (Holliday and Komatsu). [6] *If $t = 1$ and $s, n \in \mathbb{Z}^+$, then*

$$\left[\left(\sum_{k=n}^{\infty} \frac{1}{f_k(s, 1)} \right)^{-1} \right] = f_n(s, 1) - f_{n-1}(s, 1) - \delta_{\mathbb{Z} \setminus 2\mathbb{Z}}(n),$$

and

$$\left[\left(\sum_{k=n}^{\infty} \frac{1}{f_k(s, 1)^2} \right)^{-1} \right] = s f_n(s, 1) f_{n-1}(s, 1) - \delta_{2\mathbb{Z}}(n).$$

According to [1], Holliday and Komatsu asked whether the above result could be generalized for other t . The following result from [1] generalizes the first sum and also considers a more general class of sums than those considered in [6].

Theorem 17 (ACMS). [1] *If $s \geq t \geq 1$ and $n, r \in \mathbb{Z}^+$ then*

$$\left[\left(\sum_{k=n}^{\infty} \frac{1}{f_{rk}(s, t)} \right)^{-1} \right] = f_{rn}(s, t) - f_{r(n-1)}(s, t) - \delta_{2\mathbb{Z}}(r(n-1)).$$

If $t = 1$ and $s, n, r \in \mathbb{Z}^+$ then

$$\left[\left(\sum_{k=n}^{\infty} \frac{1}{f_{rk}(s, 1)^2} \right)^{-1} \right] = f_{rn}(s, 1)^2 - f_{r(n-1)}(s, 1)^2 - \delta_{2\mathbb{Z}}(r(n-1)).$$

It is conjectured that there is an analogue of this theorem replacing t with $-t$.

Conjecture 2 (ACMS). [1] *If $s > t \geq 1$ with $(s, -t) \neq (2, -1)$ and $n, r \in \mathbb{Z}^+$ then*

$$\left[\left(\sum_{k=n}^{\infty} \frac{1}{f_{rk}(s, -t)} \right)^{-1} \right] = f_{rn}(s, -t) - f_{r(n-1)}(s, -t) - 1.$$

If $t = -1$ and $s, n, r \in \mathbb{Z}^+$ then

$$\left[\left(\sum_{k=n}^{\infty} \frac{1}{f_{rk}(s, -1)^2} \right)^{-1} \right] = f_{rn}(s, -1)^2 - f_{r(n-1)}(s, -1)^2 - 1.$$

We show that the conjecture holds for sufficiently large s, t .

Theorem 18. *Let $s > t \geq 1$ be a pair of integers with $(s, -t) \neq (2, -1)$ and $n, r \in \mathbb{Z}^+$. If s, t are sufficiently large, we have that*

$$\left[\left(\sum_{k=n}^{\infty} \frac{1}{f_{rk}(s, -t)} \right)^{-1} \right] = f_{rn}(s, -t) - f_{r(n-1)}(s, -t) - 1.$$

Let $t = -1$ and $s, n, r \in \mathbb{Z}^+$. If s is sufficiently large, we have that

$$\left[\left(\sum_{k=n}^{\infty} \frac{1}{f_{rk}(s, -1)^2} \right)^{-1} \right] = f_{rn}(s, -1)^2 - f_{r(n-1)}(s, -1)^2 - 1.$$

We will use the following result in [1] in the proof of this theorem.

Lemma 5 (ACMS). [1] *Let $r, m, n \in \mathbb{P}$ and s, t be arbitrary integers. We have $f_{rn}(s, t)f_{r(n+m-1)}(s, t) - f_{r(n-1)}(s, t)f_{r(n+m)}(s, t) = (-t)^{r(n-1)}f_r(s, t)f_{rm}(s, t)$.*

Now we begin the proof of Theorem 18.

Proof. We will first show that each of the two series $\sum_{k=n}^{\infty} \frac{1}{f_{rk}(s, -t)}$ and $\sum_{k=n}^{\infty} \frac{1}{f_{rk}(s, -1)^2}$ converges under the given conditions. By Proposition 1.1 of [1], we have that $f_n(s, -t) = \frac{X^n - Y^n}{X - Y}$, where $X = \frac{s + \sqrt{s^2 - 4t}}{2}$ and $Y = \frac{s - \sqrt{s^2 - 4t}}{2}$. This means that the ratio of consecutive terms of the sum $\sum_{k=n}^{\infty} \frac{1}{f_{rk}(s, -t)}$ is of the form $\frac{X^p - Y^p}{X^{p+r} - Y^{p+r}}$ for some $p \geq rn$. Similarly, we have that the ratio of consecutive terms of the sum $\sum_{k=n}^{\infty} \frac{1}{f_{rk}(s, -1)^2}$ is of the form $\left(\frac{X^p - Y^p}{X^{p+r} - Y^{p+r}} \right)^2$ for some $p \geq rn$. Note that $\lim_{p \rightarrow \infty} \frac{X^p - Y^p}{X^{p+r} - Y^{p+r}} = \frac{1}{X^r}$. Since $X > 1$ for all pairs $(s, -t) \neq (2, -1)$, it follows

that the two series $\sum_{k=n}^{\infty} \frac{1}{f_{rk}(s, -t)}$ and $\sum_{k=n}^{\infty} \frac{1}{f_{rk}(s, -1)^2}$ both converge.

We now examine the sum $\sum_{k=n}^{\infty} \frac{1}{f_{rk}(s, -t)}$. Let $B(n) = \sum_{k=n}^{\infty} \frac{1}{f_{rk}(s, -t)}$. Note that

$$\left[\left(\sum_{k=n}^{\infty} \frac{1}{f_{rk}(s, -t)} \right)^{-1} \right] = f_{rn}(s, -t) - f_{r(n-1)}(s, -t) - 1 \text{ if and only if } f_{rn}(s, -t) - f_{r(n-1)}(s, -t) - 1 \leq \frac{1}{B(n)} < f_{rn}(s, -t) - f_{r(n-1)}(s, -t).$$

The proof that $\frac{1}{B(n)} < f_{rn}(s, -t) - f_{r(n-1)}(s, -t)$ is very similar to the corresponding proof in Theorem 2 for $f_n(s, t)$ (see [1]). It suffices to show that $1 < B(n)(f_{rn}(s, -t) - f_{r(n-1)}(s, -t))$. Note that the first term of the product $B(n)f_{rn}(s, -t)$ is equal to 1, which we subtract from both sides of the inequality. Now we will compare the term for $k = n + m$ ($m \geq 1$) in $f_{rn}(s, -t)B(n)$ with the term for $k = n + m - 1$ in $f_{r(n-1)}(s, -t)B(n)$ and see that it suffices to show that

$$0 < \frac{f_{rn}(s, -t)}{f_{r(n+m)}(s, -t)} - \frac{f_{r(n-1)}(s, -t)}{f_{r(n+m-1)}(s, -t)}.$$

By Lemma 1, we have that

$$f_{rn}(s, -t)f_{r(n+m-1)}(s, -t) - f_{r(n-1)}(s, -t)f_{r(n+m)}(s, -t) = t^{r(n-1)}f_r(s, -t)f_{rm}(s, -t).$$

Dividing by $f_{r(n+m-1)}(s, -t)f_{r(n+m)}(s, -t)$, we have that

$$\frac{f_{rn}(s, -t)}{f_{r(n+m)}(s, -t)} - \frac{f_{r(n-1)}(s, -t)}{f_{r(n+m-1)}(s, -t)} = t^{r(n-1)} \frac{f_r(s, -t)f_{rm}(s, -t)}{f_{r(n+m)}(s, -t)f_{r(n+m-1)}(s, -t)}.$$

We claim that $f_k(s, -t) > 0$ for all $k \geq 1$. It suffices to show that the sequence $\{f_k(s, -t)\}$ is monotonically increasing since $f_1(s, -t) = 1$. This can be proved by induction. We have $f_1(s, -t) = 1$ and $f_2(s, -t) = s \geq 3$. Assume that $f_n \geq f_{n-1}$. Then, $f_{n+1} - f_n = (s-1)f_n - tf_{n-1} \geq (s-1)f_n - (s-1)f_{n-1} = (s-1)(f_n - f_{n-1}) \geq 0$. Thus, $f_k(s, -t) > 0$ for all $k \geq 1$. This means that

$$\frac{f_{rn}(s, -t)}{f_{r(n+m)}(s, -t)} - \frac{f_{r(n-1)}(s, -t)}{f_{r(n+m-1)}(s, -t)} = t^{r(n-1)} \frac{f_r(s, -t)f_{rm}(s, -t)}{f_{r(n+m)}(s, -t)f_{r(n+m-1)}(s, -t)} > 0.$$

As for the other bound, the same procedure as the one used in the previous paragraph can be used to show that proving this bound reduces to showing that

$$\frac{f_{rn}(s, -t)}{f_{r(n+m)}(s, -t)} - \frac{f_{r(n-1)}(s, -t)}{f_{r(n+m-1)}(s, -t)} \leq \frac{1}{f_{r(n+m-1)}(s, -t)}.$$

Cross-multiplying and using Lemma 1, we find that it suffices to show that

$$t^{r(n-1)} \leq \frac{1}{f_r(s, -t)} \frac{f_{r(n+m)}(s, -t)}{f_{rm}(s, -t)}$$

. In [1], it was shown that

$$t^{r(n-1)} \leq \frac{1}{t} \frac{1}{f_r(s, t)} \frac{f_{r(n+m)}(s, t)}{f_{rm}(s, t)} \leq \frac{1}{t} \frac{1}{f_r(s, -t)} \frac{f_{r(n+m)}(s, t)}{f_{rm}(s, t)}$$

. Since $\frac{1}{f_r(s, -t)} \geq \frac{1}{f_r(s, t)}$ for all $r, s, t \in \mathbb{Z}^+$, it suffices to show that

$$\frac{1}{t} \frac{f_{r(n+m)}(s, t)}{f_{rm}(s, t)} \leq \frac{f_{r(n+m)}(s, -t)}{f_{rm}(s, -t)}$$

for sufficiently large s, t . Let $X_1 = \frac{s+\sqrt{s^2-4t}}{2}$, $Y_1 = \frac{s-\sqrt{s^2-4t}}{2}$, $X_2 = \frac{s+\sqrt{s^2+4t}}{2}$, and $Y_2 = \frac{s-\sqrt{s^2+4t}}{2}$. Rewriting the inequality in terms of the X_i and Y_i , we claim that

$$\frac{1}{t} \frac{X_2^{r(n+m)} - Y_2^{r(n+m)}}{X_2^{rm} - Y_2^{rm}} \leq \frac{X_1^{r(n+m)} - Y_1^{r(n+m)}}{X_1^{rm} - Y_1^{rm}}$$

for sufficiently large s, t . We have

$$\frac{1}{t} \frac{X_2^{r(n+m)} - Y_2^{r(n+m)}}{X_2^{rm} - Y_2^{rm}} \leq \frac{X_1^{r(n+m)} - Y_1^{r(n+m)}}{X_1^{rm} - Y_1^{rm}}$$

if and only if

$$\frac{(X_2^{r(n+m)} - Y_2^{r(n+m)})(X_1^{rm} - Y_1^{rm})}{(X_1^{r(n+m)} - Y_1^{r(n+m)})(X_2^{rm} - Y_2^{rm})} \leq t$$

. Note that $\lim_{s \rightarrow \infty} \frac{X_1}{X_2} = 1$ and $\lim_{s \rightarrow \infty} \frac{Y_1}{Y_2} = -1$. We obtain the first limit as follows. Since $s > t$, we have $\frac{s+\sqrt{s^2-4s}}{s+\sqrt{s^2+4s}} < \frac{X_1}{X_2} < 1$. Since $\lim_{s \rightarrow \infty} \frac{s+\sqrt{s^2-4s}}{s+\sqrt{s^2+4s}} = 1$, we have $\lim_{s \rightarrow \infty} \frac{s+\sqrt{s^2-4t}}{s+\sqrt{s^2+4t}} = 1$ for all $t < s$. We also have that $\lim_{s \rightarrow \infty} \frac{Y_i}{X_i} = 0$. Rewriting

$$\frac{(X_2^{r(n+m)} - Y_2^{r(n+m)})(X_1^{rm} - Y_1^{rm})}{(X_1^{r(n+m)} - Y_1^{r(n+m)})(X_2^{rm} - Y_2^{rm})}$$

in terms of $\frac{X_2}{X_1}$, $\frac{Y_2}{Y_1}$, and $\frac{Y_i}{X_i}$, we have the following.

$$\begin{aligned} & \frac{(X_2^{r(n+m)} - Y_2^{r(n+m)})(X_1^{rm} - Y_1^{rm})}{(X_1^{r(n+m)} - Y_1^{r(n+m)})(X_2^{rm} - Y_2^{rm})} \\ &= \frac{\left(1 - \left(\frac{Y_2}{X_2}\right)^{r(n+m)}\right) \left(1 - \left(\frac{Y_1}{X_1}\right)^{rm}\right)}{\left(\left(\frac{X_1}{X_2}\right)^{r(n+m)} - \left(\frac{Y_2}{X_2}\right)^{r(n+m)} \left(\frac{Y_1}{Y_2}\right)^{r(n+m)}\right) \left(\left(\frac{X_2}{X_1}\right)^{rm} - \left(\frac{Y_1}{X_1}\right)^{rm} \left(\frac{Y_2}{Y_1}\right)^{rm}\right)} \end{aligned}$$

Using the limits given above, we find that

$$\lim_{s \rightarrow \infty} \frac{(X_2^{r(n+m)} - Y_2^{r(n+m)})(X_1^{rm} - Y_1^{rm})}{(X_1^{r(n+m)} - Y_1^{r(n+m)})(X_2^{rm} - Y_2^{rm})} = 1$$

. Thus, the inequality

$$\frac{(X_2^{r(n+m)} - Y_2^{r(n+m)})(X_1^{rm} - Y_1^{rm})}{(X_1^{r(n+m)} - Y_1^{r(n+m)})(X_2^{rm} - Y_2^{rm})} \leq t$$

holds for any $t \geq 2$ for sufficiently large s .

Now we turn to the sum $\sum_{k=n}^{\infty} \frac{1}{f_{rk}(s, -1)^2}$. Let $C(n) = \sum_{k=n}^{\infty} \frac{1}{f_{rk}(s, -1)^2}$. Note that

$$\left[\left(\sum_{k=n}^{\infty} \frac{1}{f_{rk}(s, -t)^2} \right)^{-1} \right] = f_{rn}(s, -t)^2 - f_{r(n-1)}(s, -t)^2 - 1 \text{ if and only if}$$

$$f_{rn}(s, -t)^2 - f_{r(n-1)}(s, -t)^2 - 1 \leq \frac{1}{C(n)} < f_{rn}(s, -t)^2 - f_{r(n-1)}(s, -t)^2$$

The proof that $\frac{1}{C(n)} < f_{rn}(s, -1)^2 - f_{r(n-1)}(s, -1)^2$ is also very similar to the corresponding proof of Theorem 2. It suffices to show that $1 < C(n)(f_{rn}(s, -1)^2 - f_{r(n-1)}(s, -1)^2)$. Note that the first term of the product $C(n)f_{rn}(s, -1)^2$ is equal to 1, which we subtract from both sides of the inequality. Now we will compare the term for $k = n + m$ ($m \geq 1$) in $f_{rn}(s, -t)^2 C(n)$ with the term for $k = n + m - 1$ in $f_{r(n-1)}(s, -t)^2 C(n)$ and see that it suffices to show that

$$0 < \left(\frac{f_{rn}(s, -1)}{f_{r(n+m)}(s, -1)} \right)^2 - \left(\frac{f_{r(n-1)}(s, -1)}{f_{r(n+m-1)}(s, -1)} \right)^2.$$

From the proof of the first part of the theorem, we have $\frac{f_{rn}(s, -t)}{f_{r(n+m)}(s, -t)} > \frac{f_{r(n-1)}(s, -t)}{f_{r(n+m-1)}(s, -t)}$ and we can obtain the above inequality by squaring both sides and setting $t = 1$.

Consider the difference between the term for $k = n + m$ in $f_{rn}(s, -1)^2 C(n)$ and the term for $k = n + m - 1$ in $f_{r(n-1)}(s, -1)^2 C(n)$. After replacing each fraction in the inequality

$$\frac{f_{rn}(s, -1)}{f_{r(n+m)}(s, -1)} - \frac{f_{r(n-1)}(s, -1)}{f_{r(n+m-1)}(s, -1)} \leq \frac{1}{f_{r(n+m-1)}(s, -1)}$$

with its square and clearing denominators, we have

$$f_{rn}(s, -1)^2 f_{r(n+m-1)}(s, -1)^2 - f_{r(n-1)}(s, -1)^2 f_{r(n+m)}(s, -1)^2 \leq f_{r(n+m-1)}(s, -1)^2.$$

If this inequality is satisfied for all $m \in \mathbb{Z}^+$, then the difference between the term for $k = n + m$ in $f_{rn}(s, -1)^2 C(n)$ and the term for $k = n + m - 1$ in $f_{r(n-1)}(s, -1)^2 C(n)$ is always positive. If it is not satisfied for any $m \in \mathbb{Z}^+$, this

difference is always negative. Let $X = \frac{s+\sqrt{s^2-4}}{2}$ and $Y = \frac{s-\sqrt{s^2-4}}{2}$. Note that $\lim_{s \rightarrow \infty} \frac{Y}{X} = 0$.

By Lemma 1, we have that

$$f_{rn}(s, -1)^2 f_{r(n+m-1)}(s, -1)^2 - f_{r(n-1)}(s, -1)^2 f_{r(n+m)}(s, -1)^2 \leq f_{rn+rm}(s, -1)^2$$

if and only if

$$f_r(s, -1) f_{rm}(s, -1) (f_{rn}(s, -1) f_{r(n+m-1)}(s, -1) + f_{r(n-1)}(s, -1) f_{r(n+m)}(s, -1)) \leq f_{rn+rm}(s, -1)^2$$

. This holds if and only if

$$\frac{1}{f_{rn}(s, -1) f_{r(n+m-1)}(s, -1) + f_{r(n-1)}(s, -1) f_{r(n+m)}(s, -1)} \frac{f_{rn+rm}(s, -1)}{f_r(s, -1)} \frac{f_{rn+rm}(s, -1)}{f_{rm}(s, -1)} \geq 1.$$

We have that $f_n(s, -1) = \frac{X^n - Y^n}{X - Y}$ [1]. Since $\lim_{s \rightarrow \infty} \frac{Y}{X} = 0$, we have $\lim_{s \rightarrow \infty} f_n(s, -1) = X^{n-1}$. This means that

$$\begin{aligned} \lim_{s \rightarrow \infty} \frac{1}{f_{rn}(s, -1) f_{r(n+m-1)}(s, -1) + f_{r(n-1)}(s, -1) f_{r(n+m)}(s, -1)} \frac{f_{rn+rm}(s, -1)}{f_r(s, -1)} \frac{f_{rn+rm}(s, -1)}{f_{rm}(s, -1)} \\ = \frac{X^{2rn+2rm-r-rm}}{X^{rn-1} X^{r(n+m-1)-1} + X^{r(n-1)-1} X^{r(n+m)-1}} \\ = \frac{X^{2rn+2rm-r-rm}}{X^{rn-1+rn+rm-r-1} + X^{rn-r-1+rn+rm-1}} \\ = \frac{X^{2rn+rm-r}}{2X^{2rn+rm-r-2}} \\ = \frac{X^2}{2}. \end{aligned}$$

Since $X^2 > 2$ for sufficiently large s , we have

$$\frac{1}{f_r(s, -1) f_{rm}(s, -1) (f_{rn}(s, -1) + f_{r(n+m-1)}(s, -1))} \frac{f_{rn+rm}(s, -1)}{f_r(s, -1)} \frac{f_{rn+rm}(s, -1)}{f_{rm}(s, -1)} > 1$$

for sufficiently large s for all $m \in \mathbb{Z}^+$.

□

4 Periodicity modulo n

Returning to periodicity of generalized Fibonacci sequences modulo n , we find a way to interpret the rank of a sequence mod p as the order of an element of the splitting field of the characteristic polynomial of the recurrence in the case where $t = -1$. We also look at possible generalizations for other t where $p \nmid t$ and the rank mod p^k .

Theorem 19. Let $X = \frac{s+\sqrt{s^2-4}}{2}$, $Y = \frac{s-\sqrt{s^2-4}}{2}$, and $D = s^2 - 4$ be the discriminant of the characteristic polynomial of the recurrence with $D \not\equiv 0 \pmod{p}$. If $(D/p) = 1$, then $r(p) = \frac{1}{2}\text{ord}(X)$ in \mathbb{F}_p^\times if $\text{ord}(X)$ is even and $r(p) = \text{ord}(X)$ if $\text{ord}(X)$ is odd. Otherwise, we have that $r(p) = \frac{1}{2}\text{ord}(X)$ in $\mathbb{F}_{p^2}^\times$ if $\text{ord}(X)$ is even and $r(p) = \text{ord}(X)$ if $\text{ord}(X)$ is odd.

Proof. Consider the matrix for the recurrence $U = \begin{pmatrix} s & t \\ 1 & 0 \end{pmatrix}$. Note that multiplying U by $\begin{pmatrix} f_n(s, -1) \\ f_{n+1}(s, -1) \end{pmatrix}$ gives $\begin{pmatrix} f_{n+1}(s, -1) \\ f_{n+2}(s, -1) \end{pmatrix}$. If $(D/p) = 1$, we can consider the eigenvalues X and Y in \mathbb{F}_p . Since $D \not\equiv 0 \pmod{p}$, the eigenvalues of the matrix U are distinct and U is diagonalizable. Thus, we can write $U = C \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} C^{-1}$ or $U = \begin{pmatrix} Y & 0 \\ 0 & X \end{pmatrix}$, where C is an invertible matrix. This means that the rank is the smallest positive integer h such that $U^h = \lambda I$ for some $\lambda \in \mathbb{F}_p$. In terms of the eigenvalues, it is the smallest positive integer h such that $X^h = Y^h$ in \mathbb{F}_p . Since X and Y are roots of $x^2 - sx + 1$, we have that $XY = 1$. So, we can rewrite $X^h = Y^h$ as $X^h = X^{-h}$ and get $X^{2h} = 1$ in \mathbb{F}_p . So, $r(p) = \frac{1}{2}\text{ord}(X)$ if $\text{ord}(X)$ is even and $r(p) = \text{ord}(X)$ if $\text{ord}(X)$ is odd.

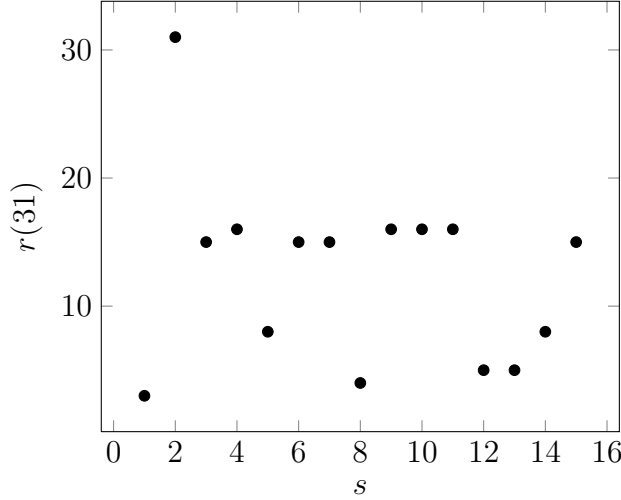
We can use a similar argument for the case where $(D/p) = -1$. However, X and Y cannot be considered in \mathbb{F}_p , so we look at the splitting field \mathbb{F}_{p^2} of the characteristic polynomial instead. Note that X and Y are also distinct in \mathbb{F}_{p^2} since $X^p = Y$ in \mathbb{F}_{p^2} and $Y \notin \mathbb{F}_p$. Using the same steps as above, we find that $r(p) = \frac{1}{2}\text{ord}(X)$ in $\mathbb{F}_{p^2}^\times$ if $\text{ord}(X)$ is even and $r(p) = \text{ord}(X)$ if $\text{ord}(X)$ is odd. \square

Remark 3. The use of the splitting field of the characteristic polynomial in the proof of this theorem is similar to its use in [4] to study periods of generalized Fibonacci sequences.

Remark 4. The rank of generalized Fibonacci sequences is a special case of the restricted period of a general linear recurrence (see [2] for a definition).

We now look at the distribution of $r(p)$ over different values of s .

Figure 1: A plot of $r(31)$ for various values of s when $t = -1$. Since the sequence starts with $0, 1, s, \dots$, the terms of the sequence mod p depend only on the residue of s mod p . In addition, only $\frac{p-1}{2}$ of the residues need to be considered since the terms of $f_n(-s, 1)$ mod p are either identical or of opposite sign, which does not change the rank of the sequence.



As we can see above for $p = 31$, there is a large amount of clustering around $\frac{p+1}{2}$ for primes p . This can be understood using the distribution of orders of elements of finite abelian groups since $r(p) = \frac{1}{2}ord(X)$ or $r(p) = ord(X)$ in \mathbb{F}_p or \mathbb{F}_{p^2} . In the case where $(D/p) = 1$, we are looking at the orders of elements of the cyclic group \mathbb{F}_p^\times . Take a generator x of \mathbb{F}_p^\times and a divisor d of $p-1$. Then, x^k is of order $\frac{p-1}{\gcd(p-1, k)}$. So, there are $\varphi(d)$ elements of order d . Since $\varphi(m) \leq \varphi(n)$ for all $m|n$, this means that there are generally more elements which are of higher order than lower order although $\varphi(n)$ does not increase monotonically. Since we have that $X^{p-1} = 1$ in \mathbb{F}_p , the order of X divides $p-1$ and the previous statement applies. In the case that $ord(X)$ is odd, $ord(X) \leq \frac{p-1}{2}$ and we can repeat the observations which we made earlier.

If $(D/p) = -1$, we can consider the order of X in $\mathbb{F}_{p^2} = \{a + bX : a, b \in \mathbb{F}_p\}$. Since $X^p = Y$ in \mathbb{F}_{p^2} and $XY = 1$, we have that $X^{p+1} = 1$ in \mathbb{F}_{p^2} and $ord(X)|p+1$. In addition, $\mathbb{F}_{p^2}^\times$ is cyclic since multiplicative subgroups of finite fields are cyclic. As above, we have that the number of elements of order d is $\varphi(d)$ and $\varphi(a) \leq \varphi(b)$ for all $a|b$. This means that the largest number of elements have order $p+1$ among elements whose orders divide $p+1$.

Making more precise statements about the rank mod p would involve looking at the order of an element in the multiplicative groups for the fields \mathbb{F}_p and \mathbb{F}_{p^2} .

more closely. In addition, we have yet to determine a relationship between s and the rank.

Since the rank mod p^k exists if and only if $p \nmid t$, it is possible to generalize the order arguments above for other t not divisible by p . The more interesting generalization relates to rank modulo general prime powers p^k . In this case we would consider the order of X in $\mathbb{Z}_{p^k}^\times$ or in $\mathbb{Z}_{p^k}[X]^\times$. Finding the order would be more complicated than for $r(p)$ since \mathbb{Z}_{p^k} is not a field for $k > 1$. However, we can still find an upper bound for $r(p^k)$ using a counting argument.

Theorem 20. *For any $s, t \in \mathbb{Z}$ with $p \nmid t$, we have that $r(p^k) \leq p^k + 1$.*

Proof. Note that we can divide the set of possible pairs of terms modulo p into equivalence classes where two pairs (a, b) and (c, d) belong to the same equivalence class if and only if $(c, d) \equiv (ka, kb) \pmod{p}$ for some k such that $p \nmid k$. We claim that all pairs of consecutive terms (f_n, f_{n+1}) with $n < r(p^k)$ belong to different equivalence classes. Assume that this is not the case. Then, there are two pairs (a, b) and (ka, kb) when considered modulo p with $k \not\equiv 0, 1 \pmod{p^k}$. Since none of the terms of the sequence between (a, b) and (ka, kb) are divisible by p^k and the terms following ka are those following a multiplied by k , there are no nonzero terms of the sequences considered modulo p^k other than $f_0(s, t)$. However, this is a contradiction since $r(p^k)$ exists. This means that there are no such pairs (a, b) and (ka, kb) before $f_{r(p^k)}$. Note that sets of pairs of the form (ka, kb) form an equivalence class. Since a single equivalence class contains $p^k - 1$ elements and each term of the sequence before f_m must come from distinct equivalence classes, we have that $r(p^k) \leq \frac{(p^k)^2 - 1}{p^k - 1} = p^k + 1$. \square

Note that it suffices to look at $r(p^k)$ in order to understand $r(n)$ for any $n \in \mathbb{N}$ by the following theorem, which we can obtain by considering the matrix U in the proof of Theorem 19.

Theorem 21 (Robinson). *[11] Let m_1, m_2 be positive integers greater than or equal to 2. Then we have that $r([m_1, m_2]) = [r(m_1), r(m_2)]$.*

It would be interesting to find a relationship between $r(p^k)$ and s for a given value of t and adapt methods used above for general linear recurrences.

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